

# ON THE RIGHT AND LEFT 4-ENGEL ELEMENTS

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**ABSTRACT.** In this paper we study left and right 4-Engel elements of a group. In particular, we prove that  $\langle a, a^b \rangle$  is nilpotent of class at most 4, whenever  $a$  is any element and  $b^{\pm 1}$  are right 4-Engel elements or  $a^{\pm 1}$  are left 4-Engel elements and  $b$  is an arbitrary element of  $G$ . Furthermore we prove that for any prime  $p$  and any element  $a$  of finite  $p$ -power order in a group  $G$  such that  $a^{\pm 1} \in L_4(G)$ ,  $a^4$ , if  $p = 2$ , and  $a^p$ , if  $p$  is an odd prime number, is in the Baer radical of  $G$ .

## 1. Introduction and Results

Let  $G$  be any group and  $n$  be a non-negative integer. For any two elements  $a$  and  $b$  of  $G$ , we define inductively  $[a, {}_n b]$ , the  $n$ -Engel commutator of the pair  $(a, b)$ , as follows:

$$[a, {}_0 b] := a, [a, b] = [a, {}_1 b] := a^{-1}b^{-1}ab \text{ and } [a, {}_n b] = [[a, {}_{n-1} b], b] \text{ for all } n > 0.$$

An element  $x$  of  $G$  is called right  $n$ -Engel if  $[x, {}_n g] = 1$  for all  $g \in G$ . We denote by  $R_n(G)$ , the set of all right  $n$ -Engel elements of  $G$ . The corresponding subset to  $R_n(G)$  which can be similarly defined is  $L_n(G)$  the set of all left  $n$ -Engel elements of  $G$  where an element  $x$  of  $G$  is called left  $n$ -Engel element if  $[g, {}_n x] = 1$  for all  $g \in G$ . A group  $G$  is called  $n$ -Engel if  $G = L_n(G)$  or equivalently  $G = R_n(G)$ . It is clear that  $R_0(G) = 1$ ,  $R_1(G) = Z(G)$  the center of  $G$ , and by a result of Kappe [4],  $R_2(G)$  is a characteristic subgroup of  $G$ . Also we have  $L_0(G) = 1$ ,  $L_1(G) = Z(G)$  and it can be easily seen that

$$L_2(G) = \{x \in G \mid \langle x \rangle^G \text{ is abelian}\},$$

where  $\langle x \rangle^G$  denotes the normal closure of  $x$  in  $G$ . In [1] it is shown that

$$L_3(G) = \{x \in G \mid \langle x, x^y \rangle \in \mathcal{N}_2 \text{ for all } y \in G\},$$

where  $\mathcal{N}_2$  is the class of nilpotent groups of class at most 2. Also it is proved that  $\langle x, y \rangle$  is nilpotent of class at most 4 whenever  $x, y \in L_3(G)$ . Newell [6] has shown that the normal closure of every element of  $R_3(G)$  in  $G$  is nilpotent of class at most 3. This shows that  $R_3(G) \subseteq \text{Fit}(G)$ , where  $\text{Fit}(G)$  is the Fitting subgroup of  $G$ , and in particular it is contained in  $B(G) \subseteq \text{HP}(G)$  where  $B(G)$  and  $\text{HP}(G)$  are the Baer radical and Hirsch-Plotkin radical of  $G$ , respectively. It is clear that

$$R_0(G) \subseteq R_1(G) \subseteq R_2(G) \subseteq \cdots \subseteq R_n(G) \subseteq \cdots$$

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2000 *Mathematics Subject Classification.* 20F45; 20F12.

*Key words and phrases.* Right Engel elements; Left Engel elements; Hirsch-Plotkin radical of a group; Baer radical of a group; Fitting subgroup.

The first author's research was in part supported by a grant from IPM (No. 87200118).

Gupta and Levin [3] have shown that the normal closure of an element in a 5-Engel group need not be nilpotent (see also [12] p. 342). Therefore  $R_n(G) \not\subseteq \text{Fit}(G)$  for  $n \geq 5$ . The following question naturally arises:

**Question 1.1.** *Let  $G$  be an arbitrary group. What are the least positive integers  $n$ ,  $m$  and  $p$  such that  $R_n(G) \not\subseteq \text{Fit}(G)$ ,  $R_m(G) \not\subseteq B(G)$  and  $R_p(G) \not\subseteq \text{HP}(G)$ ?*

To find integer  $n$  in Question 1.1 we have to answer the following.

**Question 1.2.** *Let  $G$  be an arbitrary group. Is it true that  $R_4(G) \subseteq \text{Fit}(G)$ ?*

Although in [1] it is shown that there exists  $n \in \mathbb{N}$  such that  $L_n(G) \not\subseteq \text{HP}(G)$ , the following question is still open.

**Question 1.3.** *Let  $G$  be an arbitrary group. What is the least positive integer  $k$  such that  $L_k(G) \not\subseteq \text{HP}(G)$ ?*

In this paper we study right and left 4-Engel elements. Our main results are the following.

**Theorem 1.4.** *Let  $G$  be any group. If  $a \in G$  and  $b^{\pm 1} \in R_4(G)$ , then  $\langle a, a^b \rangle$  is nilpotent of class at most 4.*

**Theorem 1.5.** *Let  $G$  be an arbitrary group and  $a^{\pm 1} \in L_4(G)$ . Then  $\langle a, a^b \rangle$  is nilpotent of class at most 4 for all  $b \in G$ .*

**Theorem 1.6.** *Let  $G$  be a group and  $a^{\pm 1} \in L_4(G)$  be a  $p$ -element for some prime  $p$ . Then*

- (1) *If  $p = 2$  then  $a^4 \in B(G)$ .*
- (2) *If  $p$  is an odd prime, then  $a^p \in B(G)$ .*

In [10, 11] Traustason has proved the above results for a 4-Engel group  $G$  (in which all elements are simultaneously right and left 4-Engel). The proofs of Theorems 1.4, 1.5 and 1.6 are somehow inspired by the arguments of [10, 11].

In Section 4, we will show that in Theorem 1.4, we cannot remove the condition  $b^{-1} \in R_4(G)$  and that in Theorems 1.5 and 1.6, the condition  $a^{-1} \in L_4(G)$  is an necessary condition.

Macdonald [5] has shown that the inverse or square of a right 3-Engel element need not be right 3-Engel. In Section 4 the GAP [2] implementation of Werner Nickel's nilpotent quotient algorithm [7] is used to prove that the inverse of a right (left, respectively) 4-Engel element is not necessarily a right (left, respectively) 4-Engel element.

## 2. Right 4-Engel elements

In this section we prove Theorem 1.4.

**Lemma 2.1.** *Let  $G$  be a group with elements  $a, b$ . Let  $x = a^b$ . We have*

- (a)  *$[b^{-1}, a, a, a, a] = 1$  if and only if  $[x^{-1}, x^a]$  commutes with  $x^a$ .*
- (b)  *$[b^{-1}, a^\tau, a^\tau, a^\tau, a^\tau] = 1$  for  $\tau = 1$  and  $\tau = -1$  if and only if  $\langle x^a, x \rangle$  is nilpotent of class at most 2.*
- (c)  *$[b^\epsilon, a^\tau, a^\tau, a^\tau, a^\tau] = 1$  for all  $\epsilon, \tau \in \{-1, +1\}$  if and only if  $\langle a, [a^b, a] \rangle$  and  $\langle a^b, [a^b, a] \rangle$  are nilpotent of class at most 2.*

*Proof.* We use a general trick for Engel commutators, namely

$$[b^{-1},_n a] = 1 \Leftrightarrow [a^{-1},_{n-1} x] = 1 \text{ where } x = a^b.$$

Applying this trick twice gives

$$[b^{-1},_4 a] = 1 \Leftrightarrow [a^{-1},_3 x] = 1 \Leftrightarrow [x^{-1},_2 x^a] = 1.$$

This proves (a). To prove (b) note that we also have

$$[b^{-1},_4 a^{-1}] = 1 \Leftrightarrow [a,{}_3 x^{-1}] = 1 \Leftrightarrow [x,{}_2 x^{-a^{-1}}] = 1 \Leftrightarrow [x^a,{}_2 x^{-1}] = 1.$$

To show (c), it follows from (b) that  $\langle x, x^a \rangle = \langle a^b, (a^b)^a \rangle = \langle a^b, [a^b, a] \rangle$  is nilpotent of class at most 2. Also we have  $\langle a^{b^{-1}}, [a^{b^{-1}}, a] \rangle$  is nilpotent of class at most 2. Now the conjugate of the latter subgroup by  $b$ , it follows that  $\langle a, [a^b, a] \rangle$ . This completes the proof.  $\square$

As immediate corollaries it follows then that

**Corollary 2.2.** (a)  $b^\epsilon \in R_4(G)$  for both  $\epsilon = 1$  and  $\epsilon = -1$  if and only if  $\langle a, [a^b, a] \rangle$  and  $\langle a^b, [a^b, a] \rangle$  are nilpotent of class at most 2 for all  $a \in G$ .  
 (b)  $a^\epsilon \in L_4(G)$  for both  $\epsilon = 1$  and  $\epsilon = -1$  if and only if  $\langle a, [a^b, a] \rangle$  and  $\langle a^b, [a^b, a] \rangle$  are nilpotent of class at most 2 for all  $b \in G$ .

**Corollary 2.3.** Let  $G$  be a group and  $a, b \in G$  such that  $[b^\epsilon, a^\tau, a^\tau, a^\tau, a^\tau] = 1$  for all  $\epsilon, \tau \in \{-1, +1\}$ . Then

- (1)  $\langle [a^b, a] \rangle^{(a)}$  and  $\langle [a^b, a] \rangle^{(a^b)}$  are both abelian.
- (2)  $[a^b, a]^{a^2} = [a^b, a]^{2a-1}$
- (3)  $[a^b, a]^{a^{2b}} = [a^b, a]^{2a^b-1}$
- (4)  $[a^b, a]^{a^{-1}} = [a^b, a]^{-a+2}$
- (5)  $[a^b, a]^{a^{-b}} = [a^b, a]^{-a^b+2}$
- (6)  $[a^b, a]^{a^b a} = [a^b, a]^{-1+aa^b+1}$
- (7)  $[a^b, a]^{aa^b a} = [a^b, a]^{-1+2aa^b-a^b+1}$
- (8)  $[a^b, a]^{aa^{2b}} = [a^b, a]^{a^b+2aa^b-a-a^b}$

We use the following result due to Sims [9].

**Theorem 2.4.** Let  $F$  be the free group of rank 2. Then the 5-th term  $\gamma_5(F)$  of the lower central series of  $F$  is equal to  $N_5$  the normal closure of the basic commutators of weight 5.

**Remark 2.5.** Suppose that  $a, b$  are arbitrary elements of a group and let  $H = \langle a, a^b \rangle$ . Then a set of basic commutators of weight 5 on  $\{a, a^b\}$  is  $\{x_1 = [a^b, a, a, a, a], x_2 = [a^b, a, a, a, a^b], x_3 = [a^b, a, a, a^b, a^b], x_4 = [a^b, a, a^b, a^b, a^b], x_5 = [[a^b, a, a], [a^b, a]], x_6 = [[a^b, a, a^b], [a^b, a]]\}$ . Hence, by Theorem 2.4, we have  $\gamma_5(H) = \langle x_1, \dots, x_6 \rangle^H$ . From on now, we fix and use the notation  $x_1, \dots, x_6$  as the mentioned commutators.

In the following calculation, one must be careful with notation. As usual  $u^{g_1+g_2}$  is shorthand notation for  $u^{g_1}u^{g_2}$ . This means that  $u^{(g_1+g_2)(h_1+h_2)} = u^{(g_1+g_2)h_1+(g_1+g_2)h_2}$  which does not have to be equal to  $u^{g_1(h_1+h_2)+g_2(h_1+h_2)}$ . We also have that

$$u^{(g_1+g_2)(-h)} = ((u^{g_1}.u^{g_2})^{-1})^h = u^{-g_2h-g_1h}.$$

This does not have to be the same as  $u^{-g_1h-g_2h}$ .

**Lemma 2.6.** *Let  $G$  be a group and  $a, b \in G$  such that  $[b^\epsilon, a^\tau, a^\tau, a^\tau, a^\tau] = 1$  for all  $\epsilon, \tau \in \{-1, +1\}$ . Then  $[a^b, a, a, a^b, a^b] = [x, x^{aa^b}]^{x^{aa^b}}$ , where  $x = [a^b, a]$ , and  $\gamma_5(\langle a, a^b \rangle) = \langle [a^b, a, a, a^b, a^b] \rangle^{\langle a, a^b \rangle}$ .*

*Proof.* By Lemma 2.1, we have  $x_1 = x_2 = x_4 = x_5 = x_6 = 1$ . Now Remark 2.5 completes the proof of the second part. To prove the first part, by Corollary 2.3 we may write

$$\begin{aligned}
[a^b, a, a, a^b, a^b] &= [x^{(-1+a)}, a^b, a^b] \\
&= x^{(-1+a)(-1+a^b)(-1+a^b)} \\
&= x^{-aa^b+a^b-1+a-aa^b+a^b-a^{2b}+aa^{2b}} \\
&= x^{-aa^b+a^b-1+a-aa^b+1+2aa^b-a-a^b} \\
&= x^{-aa^b-1-aa^b+1+2aa^b} \\
&= x^{-aa^b} [x, x^{aa^b}] x^{aa^b}
\end{aligned}$$

□

**Proof Of Theorem 1.4.** By Corollary 2.3,  $\langle [a^b, a] \rangle^{\langle a \rangle}$  is an abelian group generated by  $[a^b, a]$  and  $[a^b, a]^a$ . As  $[a^b, a^{-1}] = [a^b, a]^{-a^{-1}}$ , the subgroup  $\langle a, [a^b, a^{-1}] \rangle$  is nilpotent of class at most 2 for all  $a \in G$ . Thus by replacing  $a$  by  $a^{-1}$  in the latter group, we have that

$$\langle a, [a^{-b}, a] \rangle = \langle a, [a^b, a]^{-a^{-b}} \rangle = \langle a, [a^b, a]^{-a^b+2} \rangle$$

is nilpotent of class at most 2 for every element  $a \in G$ . Now if  $x = [a^b, a]$ , it is enough to show that  $[x^{a^b}, x^a] = 1$ . Since  $\langle a, x^{-a^b+2} \rangle$  is nilpotent of class at most 2, we have

$$\begin{aligned}
1 &= [x^{-a^b+2}, a, a] \\
&= x^{(-a^b+2)(-1+a)(-1+a)} \\
&= x^{(-2+a^b-a^b a+2a)(-1+a)} \\
&= x^{(-3+a^b-aa^b+1+2a)(-1+a)} \\
&= x^{-2a-1+aa^b-a^b+3-3a+a^b a-aa^b a+a+2a^2} \\
&= x^{aa^b-a^b+2-3a+a^b-aa^b+3a-2} \\
&= x^{-aa^b-2+aa^b+2} x^{-a^b-3a+a^b+3a} \\
&= [x^{aa^b}, x^2] [x^{a^b}, x^{3a}].
\end{aligned}$$

Therefore  $[x^2, x^{aa^b}] = [x^{a^b}, x^{3a}]$  (1).

Also

$$\begin{aligned}
1 &= [x^{-a^b+2}, a, x^{-a^b+2}] \\
&= [x^{(-a^b+2)(-1+a)}, x^{-a^b+2}] \\
&= [x^{(-2+a^b-a^b a+2a)}, x^{-a^b+2}] \\
&= [x^{(-3+a^b-aa^b+1+2a)}, x^{-a^b+2}] \\
&= [x^{-aa^b+2a}, x^{-a^b+2}] \\
&= [x^{-aa^b}, x^{-a^b+2}]^{x^{2a}} [x^{2a}, x^{-a^b+2}] \\
&= [x^{-aa^b}, x^2] [x^{2a}, x^{-a^b}] \\
&= [x^2, x^{aa^b}] [x^{a^b}, x^{2a}].
\end{aligned}$$

$$\text{Thus } [x^2, x^{aa^b}] = [x^{a^b}, x^{2a}]^{-1} \quad (2).$$

Furthermore

$$\begin{aligned}
1 &= [x^{a^b-2}, a, a] \\
&= x^{(a^b-2)(-1+a)(-1+a)} \\
&= x^{(1-a^b+aa^b+1-2a)(-1+a)} \\
&= x^{2a-1-aa^b+a^b-1+a-a^b a+aa^b a+a-2a^2} \\
&= x^{2a-1-aa^b+a^b-1+a-1-aa^b+1-1+2aa^b-a^b+1+a-4a+2} \\
&= x^{-aa^b+a^b-2+a+aa^b-a^b+2-a} \\
&= x^{-aa^b-2+aa^b+2} x^{-a^b-a+a^b+a} \\
&= [x^{aa^b}, x^2] [x^{a^b}, x^a].
\end{aligned}$$

$$\text{Therefore } [x^2, x^{aa^b}] = [x^{a^b}, x^a] \quad (3).$$

Now by (3),  $[x^{a^b}, x^a]^{x^a} = [x^{a^b}, x^a]$  and by (1) and (3) we have  $[x^{a^b}, x^a]^3 = [x^{a^b}, x^{3a}] = [x^{a^b}, x^a]$ . Hence  $[x^{a^b}, x^a]^2 = 1$  (\*). Also by (2) and (3) we have  $[x^{a^b}, x^a]^{-2} = [x^{a^b}, x^{2a}]^{-1} = [x^{a^b}, x^a]$ . Thus  $[x^{a^b}, x^a]^3 = 1$  (\*\*). Now it follows from (\*) and (\*\*) that  $[x^{a^b}, x^a] = 1$ . This completes the proof.  $\square$

### 3. Left 4-Engel elements

In this section we prove Theorems 1.5 and 1.6. The argument of Lemma 3.1 is very much modeled on an argument given in [10].

**Lemma 3.1.** *Let  $G$  be any group. If  $a^{\pm 1} \in L_4(G)$ , then  $[x^{a^b}, x^a] = [x, x^{aa^b}]$ , where  $x = [a^b, a]$  for all  $b \in G$ .*

*Proof.* From Corollary 2.2-(b) we have  $\langle a, [a^b, a] \rangle \in \mathcal{N}_2$  for all  $b \in G$ . Thus  $\langle a, [a^{bx}, a] \rangle \in \mathcal{N}_2$ . Therefore  $[a^{bx}, a^{-1}, a, a] = 1$ . We have

$$\begin{aligned}
[a^{bx}, a] &= [x^{-1} a^b x, a] \\
&= [x^{-1} a^b, a]^x [x, a] \\
&= [x^{-1}, a]^{a^b x} [a^b, a] x^{-1} x^a \\
&= (xx^{-a})^{a^b x} x^a \\
&= x^{-1-aa^b+a^b+1+a}.
\end{aligned}$$

Let  $y = [a^{bx}, a^{-1}] = [a^{bx}, a]^{-a^{-1}}$ . Then

$$\begin{aligned} y &= x^{-1-a^{-1}-a^b a^{-1}+aa^b a^{-1}+a^{-1}} \\ &= x^{-1-a^{-1}+(a^{-1}-a^{-1}a^b-a^{-1})+(a^{-1}+a^b-a^{-1})+a^{-1}} \\ &= x^{-1+aa^b-a^b}. \end{aligned}$$

We then have  $y^{-a} = [a^{bx}, a] = x^{-1-aa^b+a^b+1+a}$  and

$$\begin{aligned} y^{a^2} &= x^{-a^2-a-a^b a+aa^b a+a} \\ &= x^{-3a+aa^b-a^b+1+a}. \end{aligned}$$

Therefore

$$\begin{aligned} 1 &= y^{-a} y y^{-a} y^{a^2} \\ &= x^{-1-aa^b+a^b+1+a} x^{-1+aa^b-a^b} x^{-1-aa^b+a^b+1+a} x^{-3a+aa^b-a^b+1+a} \\ &= x^{-1+a^b+a-a^b} x^{-1-aa^b+a^b+1-2a+aa^b-a^b+1+a} \\ &= x^{-1+a^b-1-aa^b+1-a+aa^b-a^b+1+a} \\ &= x^{-1+a^b} x^{-1-aa^b+1+aa^b} x^{-a-a^b+a+a^b} x^{-a^b+1} \\ &= x^{-1+a^b} [x, x^{aa^b}] [x^a, x^{a^b}] x^{-a^b+1}. \end{aligned}$$

Conjugation with  $x^{-1+a^b}$  gives

$$1 = [x, x^{aa^b}] [x^a, x^{a^b}].$$

□

**Proof of Theorem 1.5.** By Lemma 2.6 we have to prove that  $x_3 = 1$ . Let  $u = [x^{a^b}, x^a] = [x, x^{aa^b}]$ . By Lemma 2.6 and Lemma 3.1 we have  $x_3 = u^{x^{aa^b}} = u$ . Thus

$$\gamma_5(\langle a, a^b \rangle) = \langle u \rangle^{\langle a, a^b \rangle}.$$

Since

$$\begin{aligned} u^a &= [x^a, x^{aa^b a}] \\ &= [x^a, x^{-1+2aa^b-a^b+1}] \\ &= [x^a, x^{-a^b}] \\ &= u \end{aligned}$$

and

$$\begin{aligned} u^{a^b} &= [x^{a^{2b}}, x^{aa^b}] \\ &= [x^{2a^b-1}, x^{aa^b}] \\ &= [x^{-1}, x^{aa^b}] \\ &= u^{-1} \end{aligned}$$

we have  $\gamma_5(\langle a, a^b \rangle) = \langle u \rangle$  and

$$\begin{aligned} \gamma_6(\langle a, a^b \rangle) &= [\langle a, a^b \rangle, \gamma_5(\langle a, a^b \rangle)] \\ &= [\langle a, a^b \rangle, \langle u \rangle] \\ &= \langle u^2 \rangle. \end{aligned}$$

Also  $1 = [u, a^b, a^b] = [u^{-2}, a^b]$  and we have  $\gamma_7(\langle a, a^b \rangle) = 1$ . On the other hand

$$\begin{aligned} [[a^b, a, a, a^b], [a^b, a]] &= [x^{-a+1-a^b+aa^b}, x] \\ &= [x^{aa^b}, x] \\ &= u^{-1} \in \gamma_6(\langle a, a^b \rangle). \end{aligned}$$

Thus  $u = 1$  and this completes the proof.  $\square$

**Corollary 3.2.** *Let  $G$  be a group and  $a^{\pm 1} \in L_4(G)$ . Then  $\langle a, a^b \rangle'$  is abelian, for all  $b \in G$ .*

**Corollary 3.3.** *Let  $G$  be an arbitrary group and  $a^{\pm 1} \in L_4(G)$ . Then every power of  $a$  is also a left 4-Engel element.*

*Proof.* By Corollary 2.3, Theorem 1.5 and Corollary 3.2,

$$\langle a^b, [a^b, a], [a^b, a]^a \rangle = \langle a^b, [a^b, a], [a^b, a, a] \rangle \in \mathcal{N}_2$$

for all  $b \in G$ . It follows that  $\langle a^{ib}, [a^{ib}, a^i] \rangle \leq \langle a^b, [a^b, a], [a^b, a]^a \rangle$  and  $\langle a^{ib}, [a^{ib}, a^i] \rangle \in \mathcal{N}_2$  for all  $b \in G$  and  $i \in \mathbb{Z}$ . Now Corollary 2.2 implies that  $a^i \in L_4(G)$  for all  $i \in \mathbb{Z}$ .  $\square$

**Lemma 3.4.** *Let  $G$  be a group and  $a^{\pm 1} \in L_4(G)$ . Then for all  $b$  in  $G$  and  $r, m, n \in \mathbb{N}$  we have*

- (1)  $[a^b, a^r]^{a^{2m}+1} = [a^b, a^r]^{2a^m}$
- (2)  $[a^{nb}, a^r]^{a^{2mb}+1} = [a^{nb}, a^r]^{2a^{mb}}$
- (3) If  $a^s = 1$  then  $[a^b, a, a^s] = [a^b, a, a]^s$
- (4)  $[a^b, a]^{a^n} = [a^b, a]^{na-(n-1)}$
- (5)  $[a^b, a]^{a^{nb}} = [a^b, a]^{na^b-(n-1)}$

*Proof.* By Corollary 2.3,  $[a^b, a]^{\langle a \rangle}$  and  $[a^b, a]^{\langle a^b \rangle}$  are both abelian and

$$\langle a^{mb}, [a^{nb}, a^r] \rangle \leq \langle a^b, [a^b, a], [a^b, a]^a \rangle$$

for all  $b$  in  $G$ . Therefore both  $\langle a^m, [a^b, a^r] \rangle$  and  $\langle a^{mb}, [a^{nb}, a^r] \rangle$  are nilpotent of class at most 2, for all  $b \in G$  and  $r, m, n \in \mathbb{N}$ . Thus

$$\begin{aligned} 1 = [a^b, a^r, a^m, a^m] &= [a^b, a^r]^{(-1+a^m)^2} \\ &= [a^b, a^r]^{1-2a^m+a^{2m}}. \end{aligned}$$

This prove part (1). Part (2) is similar to part (1) and the other parts are straightforward by Corollary 2.3 and induction.  $\square$

**Lemma 3.5.** *Let  $G$  be a group and  $a^{\pm 1} \in L_4(G)$ . If  $o(a) = p^i$ , for  $p = 2$  and  $i \geq 3$  or some odd prime  $p$  and  $i \geq 2$ , then  $a^{p^{i-1}} \in L_2(G)$ .*

*Proof.* First let  $p = 2$  and  $m = p^{i-3}$ . Then  $a^{8m} = 1$  and we have so

$$\begin{aligned} 1 = [a^b, a^{8m}] &= [a^b, a^{4m}]^{1+a^{4m}} \\ &= [a^b, a^{4m}]^2. \end{aligned} \quad \text{by Lemma 3.4}$$

Now we have

$$[b, a^{p^{i-1}}, a^{p^{i-1}}] = [b, a^{4m}, a^{4m}] = [a^{4mb}, a^{4m}]^{-a^{4m(-b+1)}}.$$

But

$$\begin{aligned}
[a^{4mb}, a^{4m}] &= [a^{2mb}, a^{4m}]^{a^{2mb}+1} \\
&= [a^{2mb}, a^{4m}]^{2a^{mb}} \quad \text{by Lemma 3.4} \\
&= [a^b, a^{4m}]^{2(a^{2m-1}+\dots+a^b+1)a^{mb}} \\
&= 1.
\end{aligned}$$

This complete the proof of the lemma in this case. Now let  $p$  be an odd prime number and  $i \geq 2$ . Then we have

$$\begin{aligned}
1 &= [a^b, a^{p^i}] \\
&= [a^b, a]^{1+a+\dots+a^{p^i-1}} \\
&= [a^b, a]^{1+a+2a-1+\dots+(p^i-1)a-(p^i-2)} \quad \text{by Lemma 3.4} \\
&= [a^b, a]^{\frac{p^i(p^i-1)}{2}a-\frac{p^i(p^i-1)}{2}+p^i} \\
&= [a^b, a]^{(p^i a - p^i)\frac{p^i-1}{2}} [a^b, a]^{p^i}
\end{aligned}$$

Now since

$$1 = [a^b, a, a^{p^i}] = [a^b, a, a]^{p^i} = [a^b, a]^{p^i a - p^i} \quad \text{by Lemma 3.4 part (3)}$$

we have  $[a^b, a^{p^i}] = [a^b, a]^{p^i} = 1$ . Let  $m = p^{i-1}$ . Then

$$[b, a^m, a^m] = [a^{mb}, a^m]^{-a^{-mb+m}}$$

and by Corollary 3.2 and Lemma 3.4 we have so

$$\begin{aligned}
[a^{mb}, a^m] &= [a^b, a^m]^{(1+a^b+\dots+a^{(m-1)b})} \\
&= [a^b, a^m]^{(1+a^b+2a^b-1+\dots+(m-1)a^b-(m-2))} \\
&= [a^b, a]^{(1+a+2a-1+\dots+(m-1)a-(m-2))(1+a^b+2a^b-1+\dots+(m-1)a^b-(m-2))} \\
&= [a^b, a]^{(\frac{m(m-1)}{2}a-\frac{m(m-1)}{2}+m)(\frac{m(m-1)}{2}a^b-\frac{m(m-1)}{2}+m)} \\
&= [a^b, a]^{m^2(\frac{m-1}{2}a-\frac{m-3}{2})(\frac{m-1}{2}a^b-\frac{m-3}{2})}.
\end{aligned}$$

Now since  $p^i \mid m^2$  and  $[a^b, a]^{p^i} = 1$  we have

$$[b, a^{p^{i-1}}, a^{p^{i-1}}] = [b, a^m, a^m] = 1.$$

This complete the proof of the lemma.  $\square$

**Proof of Theorem 1.6.** Let  $p$  be a prime number and  $o(a) = p^i$ . If  $p = 2$  and  $i \leq 2$  then the assertion is obvious. Therefore let  $i \geq 3$  if  $p = 2$ ; and  $i \geq 2$  if  $p$  is an odd prime number. By Lemma 3.5 and Corollary 2.2

$$1 \trianglelefteq \langle a^{p^{n-1}} \rangle^G \trianglelefteq \langle a^{p^{n-2}} \rangle^G \trianglelefteq \dots \trianglelefteq \langle a^p \rangle^G$$

is a series of normal subgroups of  $G$  with abelian factors. This implies that  $K = \langle a^p \rangle^G$  is soluble of derived length at most  $n - 1$ . By Corollary 3.3,  $a^p$  and so all its conjugates in  $G$  belong to  $L_4(G)$  and in particular they are in  $L_4(K)$ . Now a result of Gruenberg [8, Theorem 7.35] implies that  $B(K) = K$ . Therefore  $a^p \in K \leq B(G)$ , as required.  $\square$



#### 4. Examples and questions

In this section we give some examples by using GAP nq package of Werner Nickel to show what we mentioned in the last two paragraphs of Section 1.

Let  $H$  be the largest nilpotent group generated by  $a, b$  such that  $a \in R_4(H)$ ; then  $H$  is nilpotent of class 8. On the other hand if  $K$  is the largest nilpotent group generated by  $a$  and  $b$  such that  $a^{\pm 1} \in R_4(K)$ , then  $K$  is nilpotent of class 7. Thus in an arbitrary group  $G$ ,  $a \in R_4(G)$  does not imply  $a^{-1} \in R_4(G)$ . One can check the above argument with the following GAP program:

```
LoadPackage("nq"); #nq package of Werner Nickel#
F:=FreeGroup(3);; a:=F.1;; b:=F.2;; x:=F.3;;
G:=F/[LeftNormedComm([a,x,x,x,x])];; H:=NilpotentQuotient(G,[x]);;
NilpotencyClassOfGroup(H);
G:=F/[LeftNormedComm([a,x,x,x,x]),LeftNormedComm([a^-1,x,x,x,x])];;
K:=NilpotentQuotient(G,[x]);; NilpotencyClassOfGroup(K);
```

Similar to above let  $N$  be the largest nilpotent group generated by  $a, b$  such that  $a, b \in L_4(N)$  and  $b^2 = 1$ . Also let  $S = \langle a, a^b \rangle$ . Then  $N$  is nilpotent of class 10 and  $S$  is nilpotent of class 6 but the largest nilpotent group  $M$  generated by  $a, b$  such that  $a^{\pm 1}, b \in L_4(M)$  and  $b^2 = 1$  is nilpotent of class 7. Therefore in an arbitrary group  $G$ ,  $a \in L_4(G)$  does not imply  $a^{-1} \in L_4(G)$  and  $b^{-1} \in L_4(G)$  is a necessary condition in Theorem 1.5 and 1.6. The following GAP program confirms the above argument:

```
F:=FreeGroup(3);; a:=F.1;; b:=F.2;; x:=F.3;;
G:=F/[LeftNormedComm([x,a,a,a,a]),LeftNormedComm([x,b,b,b,b]),b^2];;
N:=NilpotentQuotient(G,[x]);; NilpotencyClassOfGroup(N);
S:=Subgroup(N,[N.1,N.2^-1*N.1*N.2]);; NilpotencyClassOfGroup(S);
G:=F/[LeftNormedComm([x,a,a,a,a]),LeftNormedComm([x,b,b,b,b]),
LeftNormedComm([x,a^-1,a^-1,a^-1,a^-1]),b^2];;
M:=NilpotentQuotient(G,[x]);; NilpotencyClassOfGroup(M);
```

We end this section by proposing some questions on bounded right Engel elements in certain classes of groups.

**Question 4.1.** *Let  $n$  be a positive integer. Is there a set of prime numbers  $\pi_n$  depending only on  $n$  and a function  $f : \mathbb{N} \rightarrow \mathbb{N}$  such that the nilpotency class of  $\langle x \rangle^G$  is at most  $f(n)$  for any  $\pi'_n$ -element  $x \in R_n(G)$  and any nilpotent or finite group  $G$ ?*

**Question 4.2.** *Let  $n$  be a positive integer. Is there a set of prime numbers  $\pi_n$  depending only on  $n$  such that the set of right  $n$ -Engel elements in any nilpotent or finite  $\pi'_n$ -group forms a subgroup?*

Note that if the answers of Questions 4.1 and 4.2 are positive, the answers of the corresponding questions for residually (finite or nilpotent  $\pi'_n$ -groups) are also positive.

**Question 4.3.** *Let  $n$  and  $d$  be positive integers. Is there a function  $g : \mathbb{N} \times \mathbb{N} \rightarrow \mathbb{N}$  such that any nilpotent group generated by  $d$  right  $n$ -Engel elements is nilpotent of class at most  $g(n, d)$ ?*

**Acknowledgements.** The authors are grateful to the referee for his very helpful comments. They also thank Yaov Segev for pointing out a flaw in our proofs. The

research of the first author was supported in part by the Center of Excellence for Mathematics, University of Isfahan.

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